Volatility v Trend Risk

A technical note on estimating and forecasting with the random walk with drift

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This note discusses the technical aspects of the blog *Volatility v Trend Risk* on forecasting and simulating sample paths with the random walk with drift.

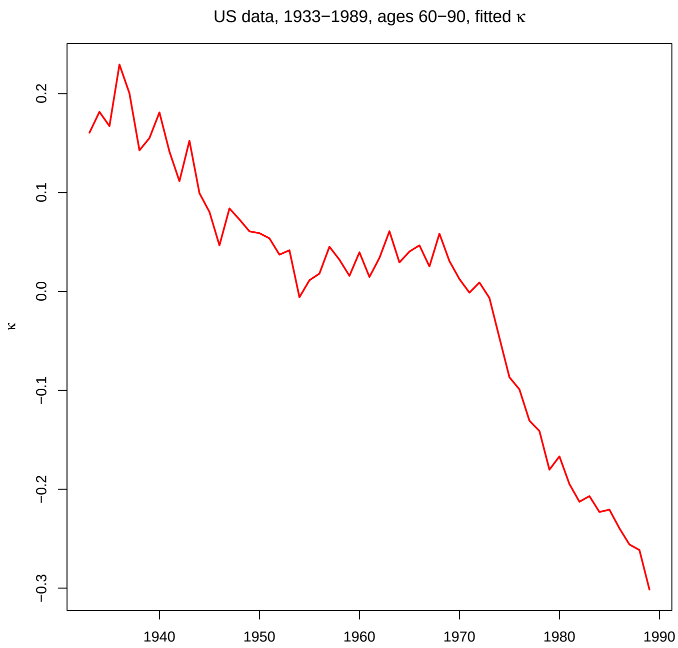

Figure 1: Fitted κ values in the Lee-Carter model for US male mortality

Suppose we have a time series $\kappa_1, \ldots, \kappa_n$; such a time series is shown in Figure 1. The random walk with drift model is

$$\kappa_j = \kappa_{j-1} + a + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \sigma^2), \quad j = 2, \ldots, n, \tag{1}$$

where the ε_j are independent. The drift and volatility parameters a and σ are to be estimated from our data. We write expression (1) in terms of $y_j = \kappa_j - \kappa_{j-1}$:

$$y_j = a + \varepsilon_j, \quad \varepsilon_j \sim \mathcal{N}(0, \sigma^2), \quad j = 2, \ldots, n \tag{2}$$

and since $y_2, \ldots, y_n$ are independent and identically distributed we immediately have

$$\hat{a} = \bar{y} = \frac{1}{n-1} \sum_2^n y_j, \quad \hat{\sigma}^2 = \frac{1}{n-2} \sum_2^n (y_j - \bar{y})^2. \tag{3}$$

The variance and standard error of $\hat{a}$ are given by

$$\text{Var}(\hat{a}) = \text{Var}(\bar{y}) = \frac{\sigma^2}{n-1} \Rightarrow \text{SE}(\hat{a}) = \frac{\hat{\sigma}}{\sqrt{n-1}}. \tag{4}$$

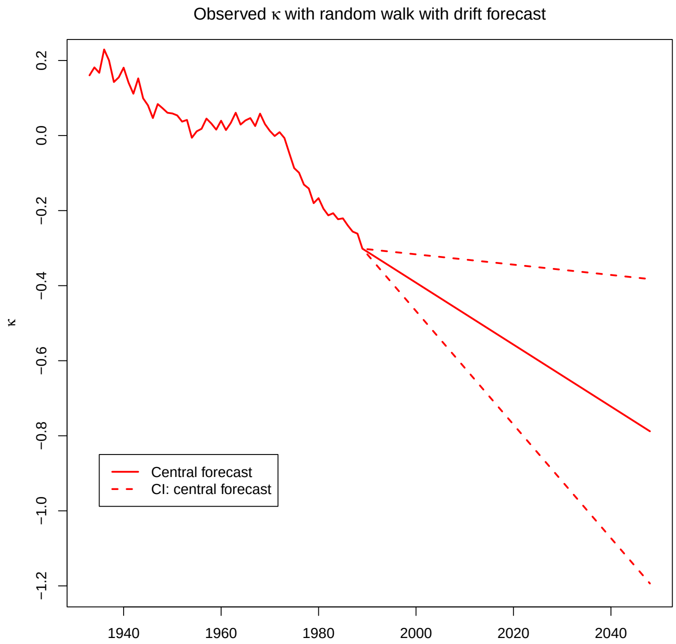

Figure 2: Fitted and forecast κ values with 95% confidence interval

A rough 95% confidence interval for a is $\hat{a} \pm 2 \times \text{SE}(\hat{a})$.

Now let's consider forecasting. The m-step ahead central forecast is

$$\hat{\kappa}^c_{n+m} = \kappa_n + m\hat{a} \tag{5}$$

with variance

$$\text{Var}(\hat{\kappa}^c_{n+m}) = m^2\text{Var}(\hat{a}) \Rightarrow \text{SE}(\hat{\kappa}^c_{n+m}) = m\frac{\hat{\sigma}}{\sqrt{n-1}} \tag{6}$$

which leads immediately to the 95% confidence interval for the central forecast shown in Figure 2.

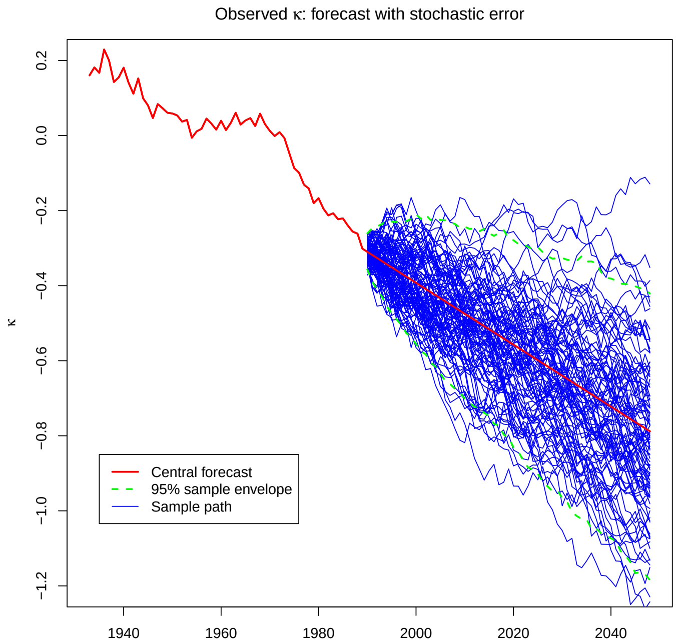

Figure 3: Sample paths for κ with stochastic error only

The actual forecast is subject to both parameter and stochastic uncertainty. We can get a feel for the relative importance of these components by isolating each in turn. We first discuss stochastic uncertainty. We fix the drift parameter at its estimated value of $\hat{a}$. Conditional on this value of a, a sample path of length s is generated by

$$\hat{\kappa}_{n+m} = \kappa_n + m\hat{a} + \sum_{1}^{m} \varepsilon_j, \quad m = 1, \ldots, s, \quad j = 1, \ldots, s \tag{7}$$

where the ε_j are independent $\mathcal{N}(0, \hat{\sigma}^2)$. Figure 3 shows the results of generating one hundred such paths. Notice that these sample paths are subject to mean reversion. The 95% sample envelope has been computed from 1000 sample paths for greater accuracy.

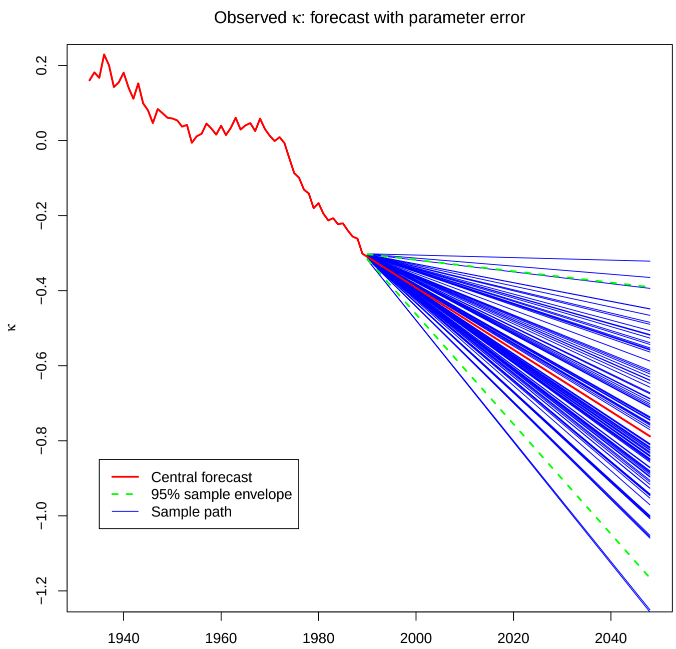

Figure 4: Sample paths for κ with parameter error only

Now let's consider parameter uncertainty. Let A be the true but unknown value of the drift parameter. Using (4) we suppose that

$$A \sim \mathcal{N}\left(\hat{a}, \frac{\hat{\sigma}^2}{n-1}\right). \tag{8}$$

We generate a future possible forecast for the mean of length s with

$$\hat{\kappa}_{n+m} = \kappa_n + mA, \quad m = 1, \ldots, s. \tag{9}$$

The result of simulating one hundred such means is shown in Figure 4. Again, the 95% sample envelope has been computed from 1000 sample paths for greater accuracy.

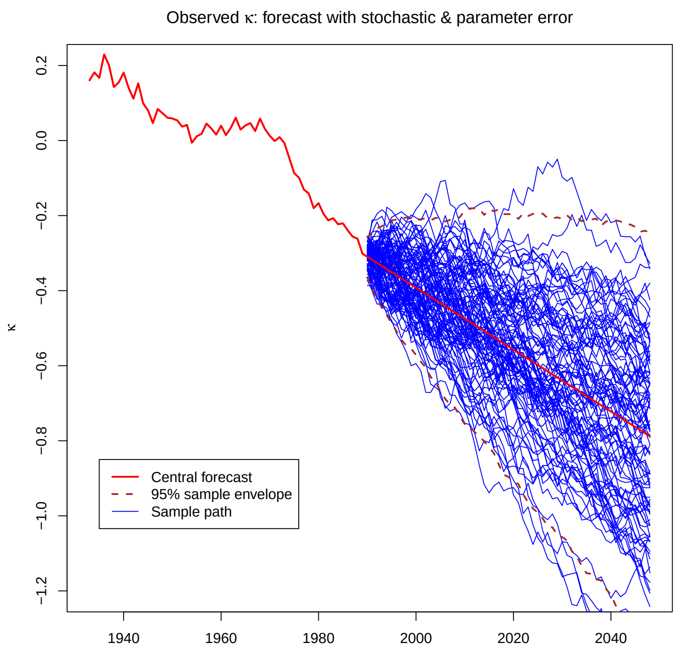

Figure 5: Sample paths for κ with both stochastic and parameter error

Figure 5 shows the results of including both stochastic and parametric uncertainty. These sample paths are generated with

$$\hat{\kappa}_{n+m} = \kappa_n + mA_i + \sum_1^m \varepsilon_{ij}, \quad m = 1, \ldots, s, \quad j = 1, \ldots, s, \quad i = 1, \ldots, 100. \qquad (10)$$

Again, the ε_{ij} are independent $\mathcal{N}(0, \hat{\sigma}^2)$ and the A_i are independently generated with (8). Once more, the 95% sample envelope has been computed from 1000 sample paths for greater accuracy.

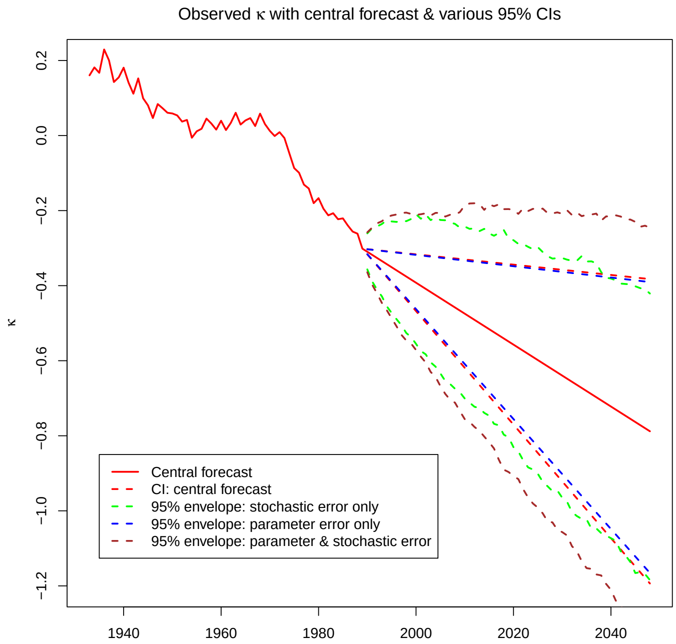

Figure 6: Central forecast for κ with various 95% sample path envelopes

How can we compare the relative contribution of stochastic and parametric uncertainty? One way is to compute the upper and lower $2\frac{1}{2}\%$ curves in each of Figures 3, 4 and 5. We do this by calculating (pointwise) the appropriate quantiles on each set of sample paths. The result is shown in Figure 6.