On Contemporary Models for Actuarial Use II: Principles

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Models: q-type and μ-type

Forfar, D.O., McCutcheon, J.J. & Wilkie, A.D. (1988). *On Graduation by Mathematical Formula*. Journal of the Institute of Actuaries, **115**, 1–149.

FMW graduated models using estimators of three parameters:

- q_x the one-year probability of death;
- μ_x the hazard rate*; or
- m_x the central rate of mortality.

* 'force of mortality' if you prefer

Models: q-type and μ-type

Two different roads to estimation:

1. q**-type models**:
   - Inspired by life table *probability* q_x.
   - Obvious statistical model Binomial ...
   - ... or is it?
2. μ**-type models**:
   - Inspired by *hazard rate* μ_x.
   - Obvious statistical model Poisson ...
   - ... or is it?

Both models have flaws, but those of the Binomial are more serious.

Models: q-type and μ-type

Flaws with simple models.

1. **q-type Model**
   - Assumes E_x persons exposed for a whole year BUT ...
   - ... some will leave before the year-end ...
   - ... while others will join part-way through the year ...
   - ... so we can't have a Binomial distribution.
2. **μ-type Model**
   - Knowing we have M individuals in the study (as we usually do) ...
   - ... the probability of more than M deaths is zero ...
   - ... so we can't have a Poisson distribution.

Models: q-type and μ-type

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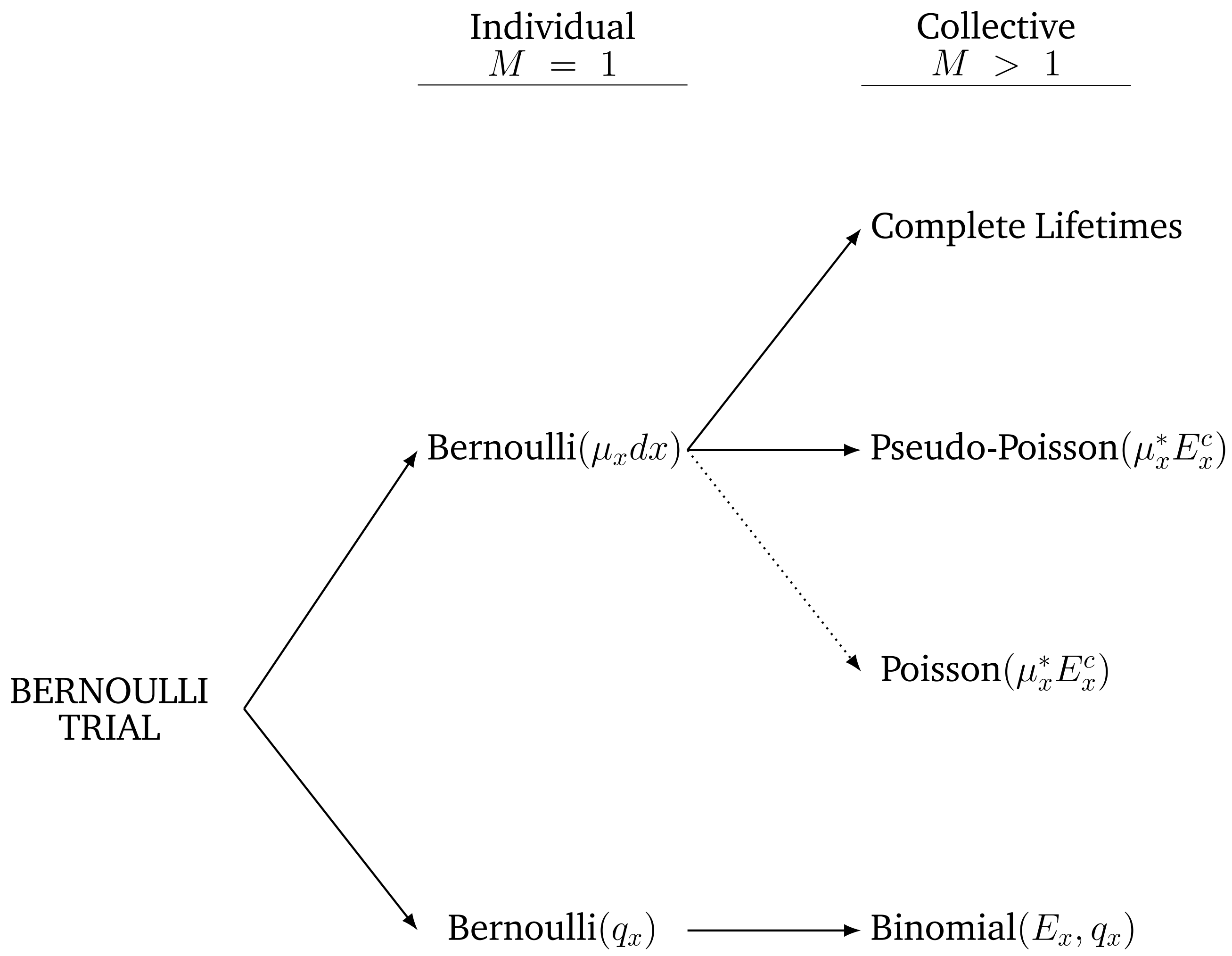
   - ... so we can't have a Poisson distribution.

'Fixing' the Binomial model leads us further into the weeds. Fixing the Poisson model leads to enlightenment!

Bernoulli Family Tree



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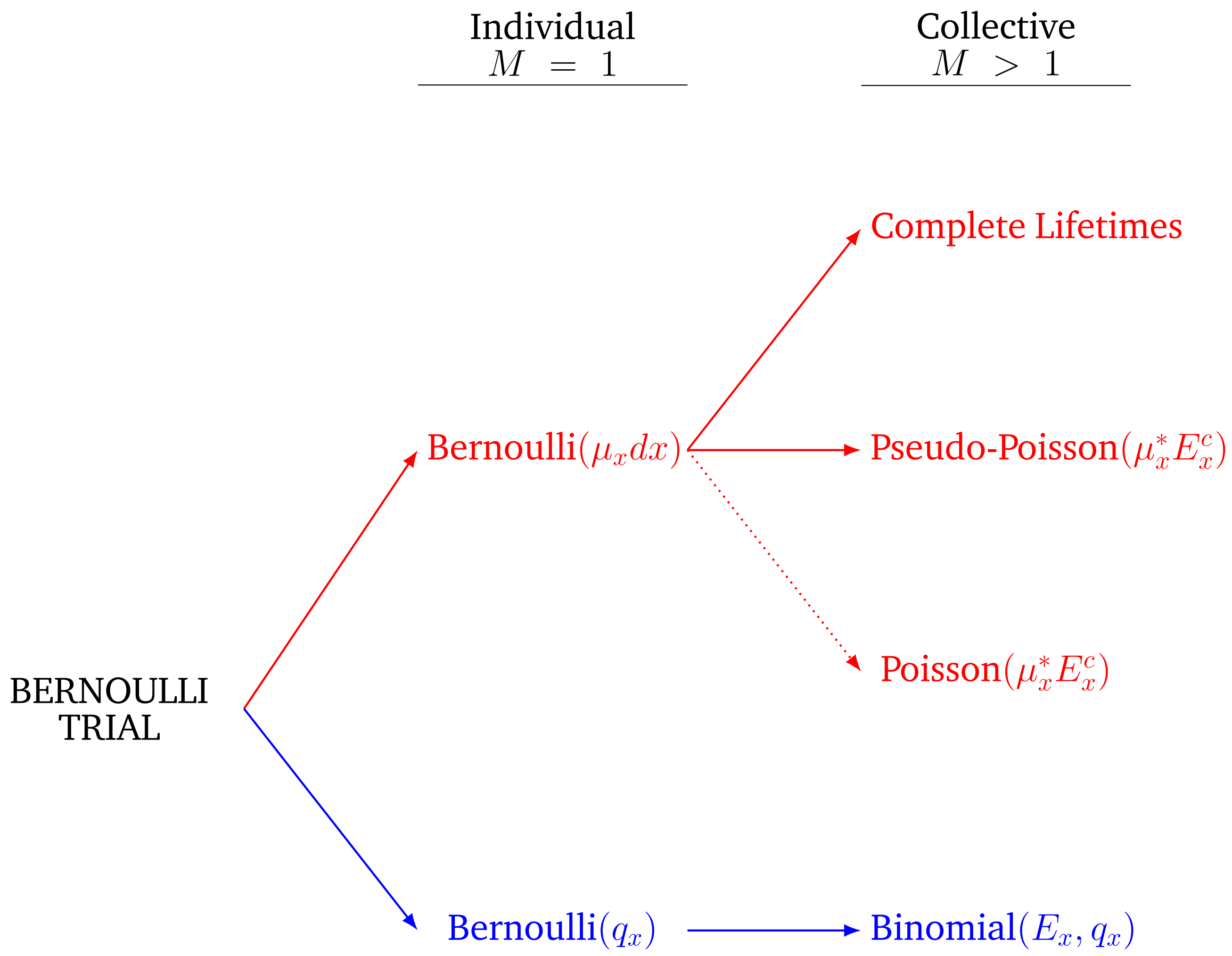

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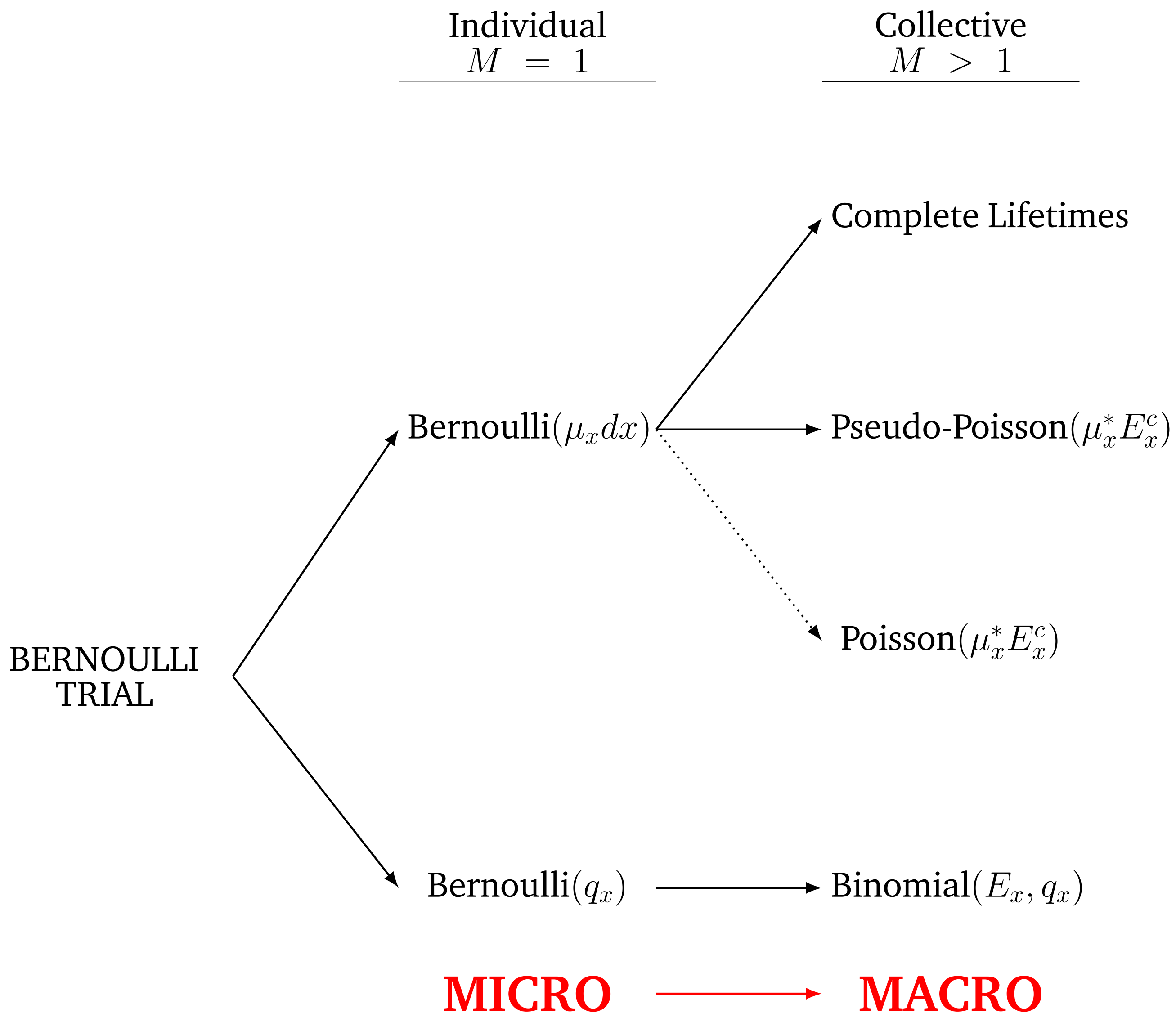

The Lower Branch: The Binomial/Bernoulli Model

Observe M lives $i = 1, 2, \ldots, M$ for one year, define 'indicator' of death d_i:

$$d_i = \begin{cases} 1 & \text{if life } i \text{ dies} \\ 0 & \text{if life } i \text{ survives} \end{cases}$$

Binomial likelihood is:

$$\begin{aligned} L_i &\propto (1 - q_x)^{M - \sum d_i} \, (q_x)^{\sum d_i} \\ &= \prod_{i=1}^{M} (1 - q_x)^{1 - d_i} \, (q_x)^{d_i}. \end{aligned}$$

. . . a product of **Bernoulli** likelihoods for each life.

... But THIS Bernoulli Model is Still a Complicated Thing!

Define T_x = random lifetime of (x) and consider $p_x = \mathrm{P}[T_x > 1]$:

Event $\{T_x > 1\}$ is highly composite:

$$
\begin{aligned}
p_x &= {}_1p_x \\
&= {}_{0.5}p_x \times {}_{0.5}p_{x+0.5} \\
&= {}_{0.25}p_x \times {}_{0.25}p_{x+0.25} \times {}_{0.5}p_{x+0.5} \\
&= {}_{0.125}p_x \times {}_{0.125}p_{x+0.125} \times {}_{0.25}p_{x+0.25} \times {}_{0.5}p_{x+0.5} \cdots \\
&= \ldots \text{and so on, } \textit{ad infinitum}.
\end{aligned}
$$

(Apologies to Zeno!)

In fact, event $\{T_x > 1\}$ is infinitely composite. Survival happens from moment to moment. And $q_x = \mathrm{P}[T_x \leq 1]$ is worse.

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1. Discrete time $\Rightarrow$ complicated events!
2. Breaking down events — the Bernoulli 'atom'
3. Building up events — the product integral
4. Data — the stochastic switch $Y(t)$
   - Survival models
   - Pseudo-Poisson models
   - True Poisson models

The Basic 'Atom' — An Infinitesimal Bernoulli Trial

The idea of the hazard rate μ_t is the infinitesimal:

$$\mathrm{P}[t < T \leq t + dt \mid T > t] = \mu_t \, dt + o(dt) \approx \mu_t \, dt.$$

For convenience (re)define the indicator:

$$d_i = \Delta N_i(t) = \begin{cases} 1 \text{ if } t < T_i < t + dt \\ 0 \text{ otherwise} \end{cases}$$

$$\mathrm{P}[\text{Obs. in } dt] \quad = \quad (1 - \mu_t \, dt)^{(1 - \Delta N_i(t))} \, (\mu_t \, dt)^{\Delta N_i(t)} = \text{Bernoulli trial}.$$

We have the infinitesimal Bernoulli trial. Not quite right yet, but let's pursue it . . .

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The Product Integral

Revision: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ or $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$.

Let $(a_i, b_i]$ be the time interval under observation by life i. Then:

$$
\begin{aligned}
\text{P[Observation}_i] &= \underbrace{\prod_{(a_i,b_i]} (1-\mu_t\,dt)^{(1-\Delta N_i(t))}(\mu_t\,dt)^{\Delta N_i(t)}}_{\text{Product Integral}} \\
&= \left(\prod_{(a_i,b_i]} (1-\mu_t\,dt)^{(1-\Delta N_i(t))}\right) \times (\mu_{b_i}\,dt)^{\Delta N_i(b_i)} \\
&= \exp\left(-\int_{a_i}^{b_i} \mu_t\,dt\right)(\mu_{b_i}\,dt)^{\Delta N_i(b_i)}.
\end{aligned}
$$

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Data: The Stochastic Switch $Y(t)$

Define the process $Y^i(t)$:

$$Y^i(t) = \begin{cases} 1 & \text{if alive and under observation at time } t^- \\ 0 & \text{otherwise} \end{cases}$$

$Y^i(t)$ acts as a stochastic 'switch' depending on the status of (x).

For example, $Y^i(t)\,\mu_t$ is a stochastic hazard rate.

$$Y^i(t)\,\mu_t = \begin{cases} \mu_t & \text{if alive and under observation at time } t^- \\ 0 & \text{otherwise} \end{cases}$$

Data: The Product Integral Likelihood

$Y^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and under observation}\}}$.

$$(1 - Y^i(t)\,\mu_t\,dt)^{1-\Delta N_i(t)}(Y^i(t)\,\mu_t\,dt)^{\Delta N_i(t)}$$

MICRO: The ‘atom’ of all Poisson-type likelihoods:

$$\begin{aligned} L_i = \text{P}[\text{Observation}_i] &= \prod_{[0,\infty)} (1 - Y^i(t)\,\mu_t\,dt)^{(1-\Delta N_i(t))}(Y^i(t)\,\mu_t\,dt)^{\Delta N_i(t)} \\ &= \underbrace{\exp\left(-\int_0^\infty Y^i(t)\,\mu_t\,dt\right)}_{\text{P[Survival]}} \underbrace{(Y^i(b_i)\,\mu_{b_i}\,dt)^{\Delta N_i(b_i)}}_{\text{P[Death]}}. \end{aligned}$$

MACRO: Universal Poisson-type likelihood

Poisson-type Models I: Survival Models

M lives, lifetimes $T_1, T_2, \ldots, T_M$, life i observed on $(a_i, b_i]$.

μ_x^θ = Parametric hazard rate on $[0, \infty)$.

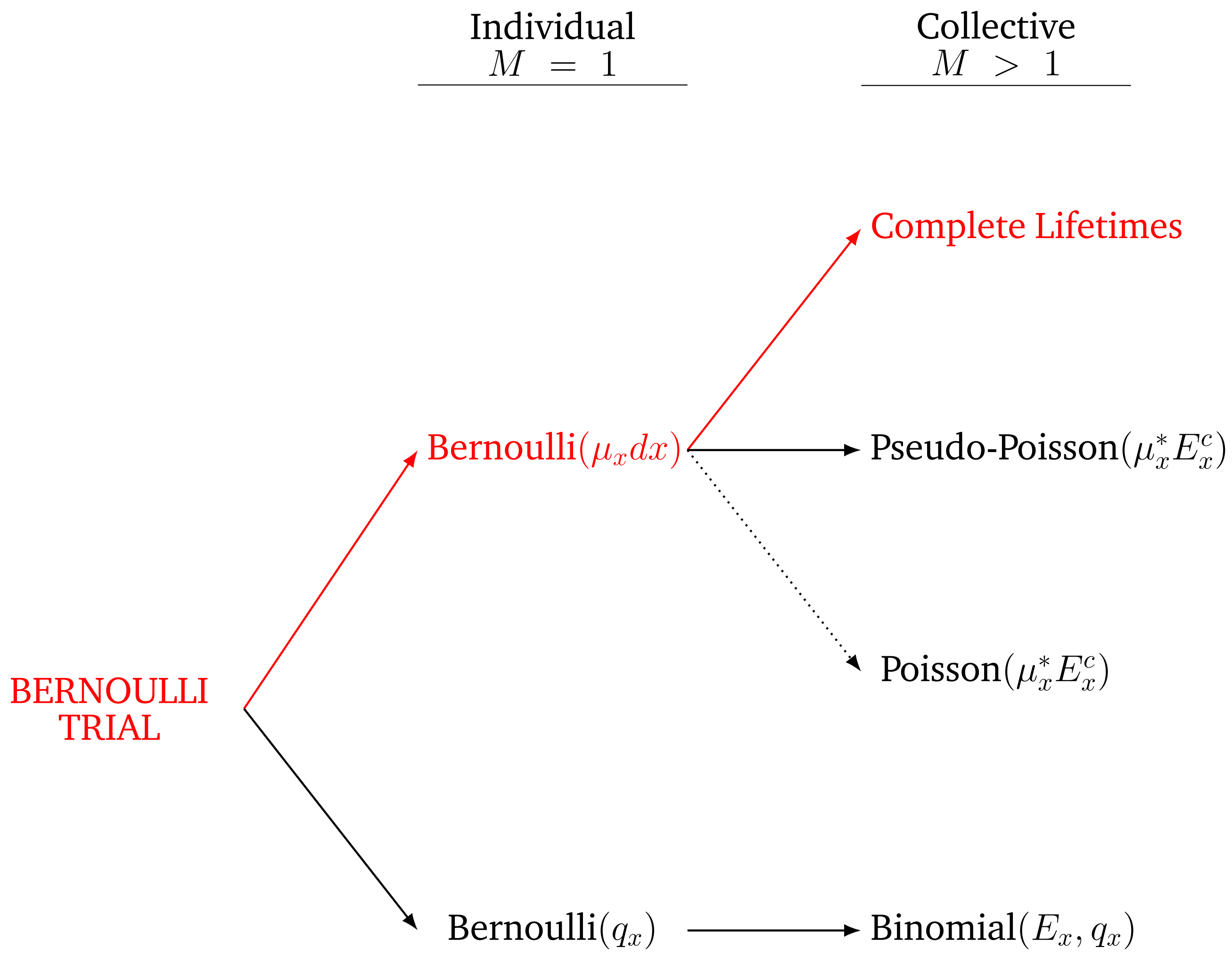
$Y^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and under observation}\}}$.

$$L = \prod_i L_i = \prod_i \prod_{[0,\infty)} (1 - Y^i(t)\, \mu_t^\theta \, dt)^{(1-\Delta N_i(t))} (Y^i(t)\, \mu_t^\theta \, dt)^{\Delta N_i(t)}$$

$$= \prod_i \exp\left(- \int_{a_i}^{b_i} Y^i(t)\, \mu_t^\theta \, dt \right) (Y^i(b_i)\, \mu_{b_i}^\theta \, dt)^{\Delta N_i(b_i)}.$$

INDIVIDUAL DATA/COMPLETE OBSERVED LIFETIMES/SURVIVAL MODEL

Bernoulli Family Tree



Poisson-type Models II: Pseudo-Poisson Models

M lives under observation, life i on $(a_i, b_i]$.

μ_x^* = Constant hazard rate on $(x, x+1]$.

$Y_x^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and recorded as `active' on } (a_i,b_i] \cap (x,x+1]\}}$.

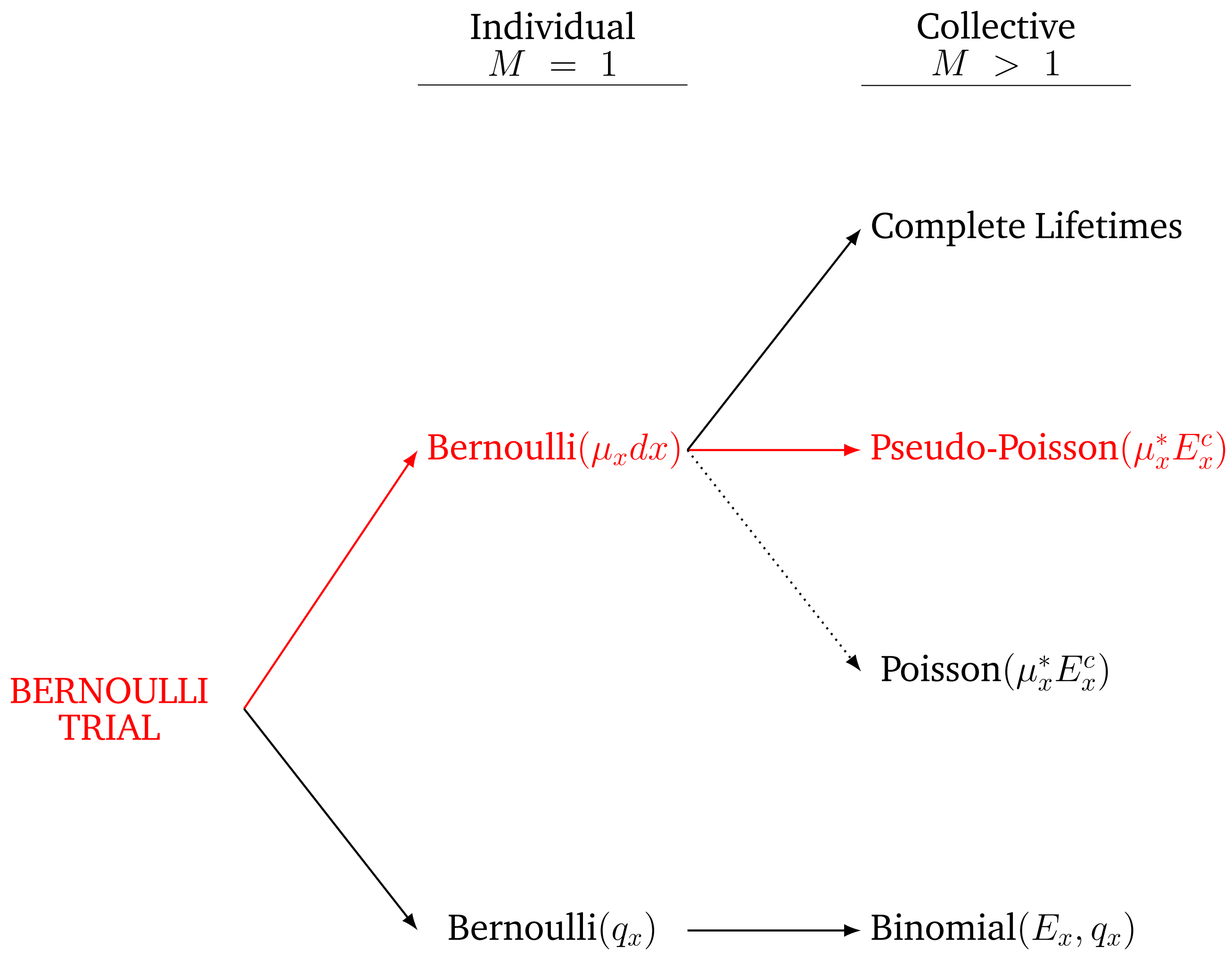
$$L = \prod_x \prod_i L^*_{x,i} = \prod_x \prod_i \prod_{[0,\infty)} (1 - Y_x^i(t)\, \mu_x^*\, dt)^{(1-\Delta N_i(t))} (Y_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)}$$

$$= \prod_x \prod_i \exp\left(- \int_x^{x+1} Y_x^i(t)\, \mu_x^*\, dt \right) \prod_{[0,\infty)} (Y_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)}$$

$$= \prod_x \exp\left(E_x^c\right) \left(\mu_x^*\right)^{D_x}.$$

M known, E_x^c random variable $\Rightarrow$ GROUPED DATA/PSEUDO-POISSON

Bernoulli Family Tree



Poisson-type Models III: True Poisson Models

Random M lives under observation, life i on $(a_i, b_i]$.

μ_x^* = Constant hazard rate on $(x, x+1]$.

$\tilde{Y}_x^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and recorded as `active' on } (a_i, b_i] \cap (x, x+1]\}}$ constrained so that E_x^c is a pre-determined constant.

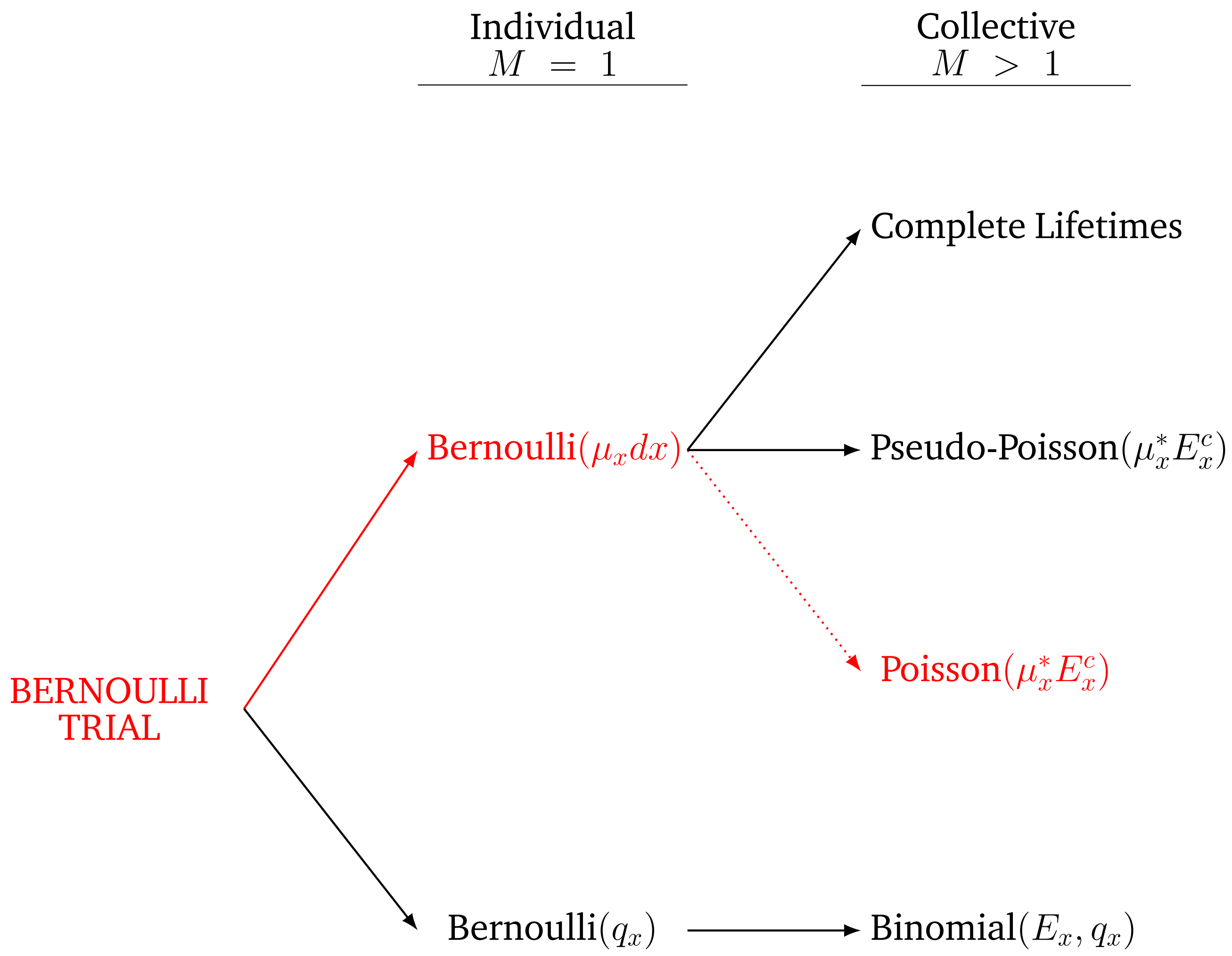
$$L = \prod_x \prod_i L_{x,i}^* \quad \propto \quad \prod_x \prod_i \prod_{[0,\infty)} (1 - \tilde{Y}_x^i(t)\, \mu_x^*\, dt)^{(1-\Delta N_i(t))} (\tilde{Y}_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)}$$

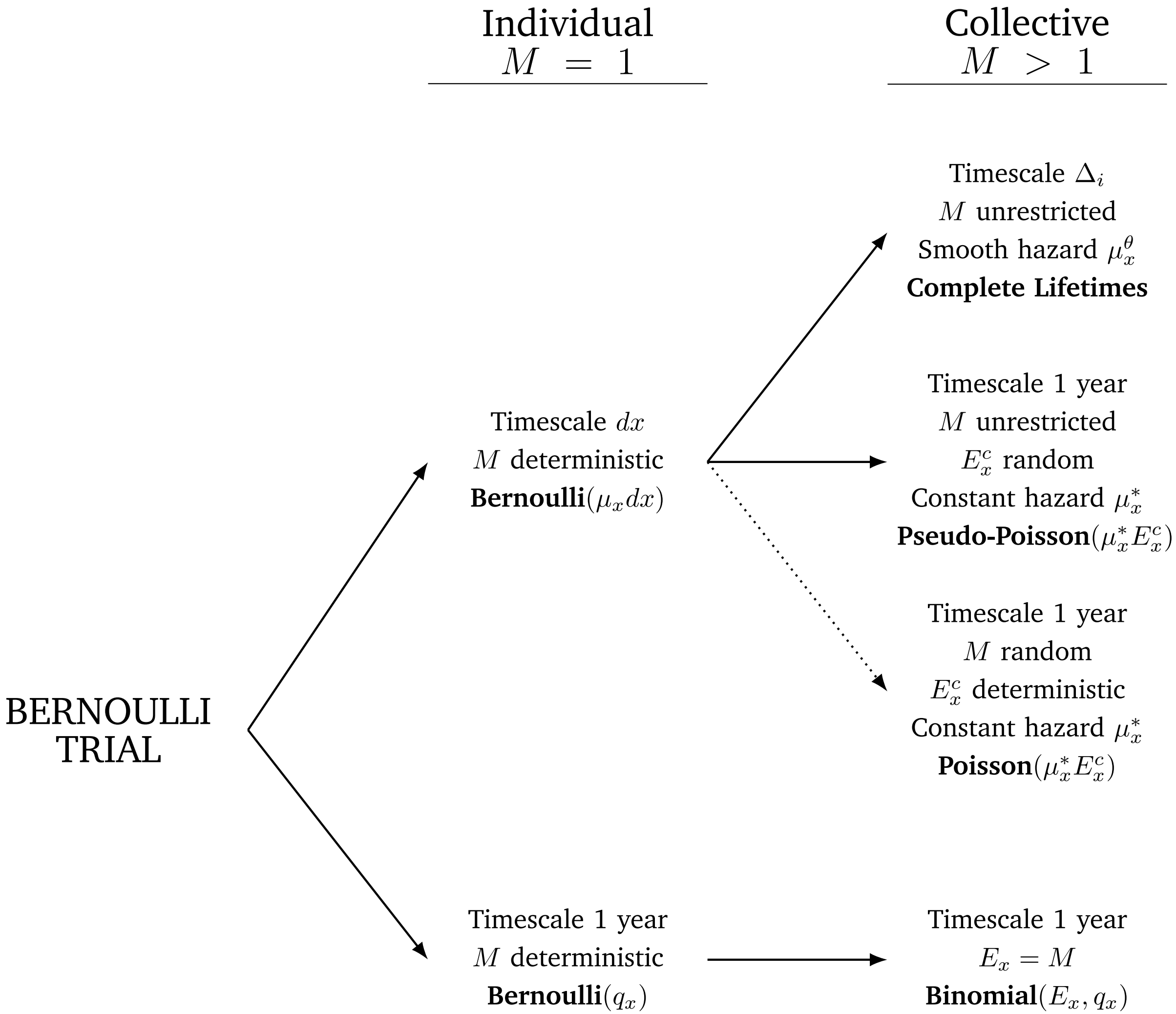
$$= \quad \prod_x \prod_i \exp\left(- \int_x^{x+1} \tilde{Y}_x^i(t)\, \mu_x^*\, dt \right) \prod_{[0,\infty)} (\tilde{Y}_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)}$$

$$= \quad \prod_x \exp\left(E_x^c\right) \left(\mu_x^*\right)^{D_x}.$$

M random variable, E_x^c known $\Rightarrow$ GROUPED DATA/TRUE POISSON

Bernoulli Family Tree




Individual
$M = 1$
Collective
$M > 1$
BERNOULLI TRIAL
Timescale dx
M deterministic
Bernoulli$(\mu_x dx)$
Timescale 1 year
M deterministic
Bernoulli(q_x)
Timescale Δ_i
M unrestricted
Smooth hazard μ_x^θ
Complete Lifetimes
Timescale 1 year
M unrestricted
E_x^c random
Constant hazard μ_x^*
Pseudo-Poisson$(\mu_x^* E_x^c)$
Timescale 1 year
M random
E_x^c deterministic
Constant hazard μ_x^*
Poisson$(\mu_x^* E_x^c)$
Timescale 1 year
$E_x = M$
Binomial(E_x, q_x)