CASS Business School, London

Mis-estimation risk: A parametric approach

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Contents

1. About the speaker and presentation

1. About the speaker

- Independent consultant on longevity risk since 2005.
- Founded longevity-related software businesses in 2006:
- Joint software venture with Heriot-Watt University in 2009:

1. About the presentation

- Presentation based on Richards (2014); printed copies available.
- Related article in November 2014 issue of *The Actuary* magazine.

2. Defining mis-estimation risk

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"[Mis-estimation risk is] the risk that the base mortality estimate is incorrect (i.e. the mortality estimate based on actual experience in the portfolio)"

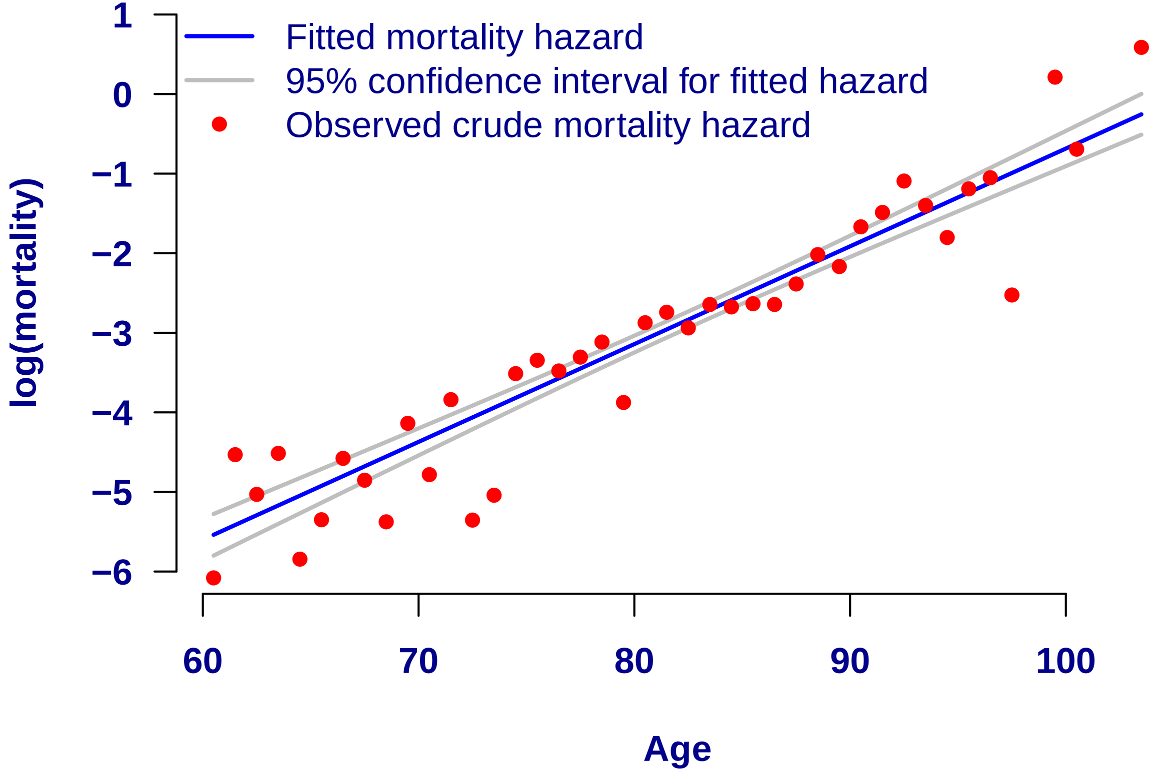
Burgess et al (2010)

"How wrong could our base mortality assumptions be, or: what if our historical experience did not reflect the underlying mortality?"

Armstrong (2013)

2. Defining mis-estimation risk

- Every statistical estimate has uncertainty:



Source: Richards (2014), Figure 3.

2. Defining mis-estimation risk

- Mis-estimation risk is the uncertainty over *current* mortality rates.
- Other risks, such as future improvements, are handled separately.
- Mis-estimation risk is statistical uncertainty due to finite data.

→ Portfolios with less data are more exposed to mis-estimation risk.

2. Mis-estimation risk

- Mis-estimation is clearly a risk for ICA and Solvency II work.
- Second-largest component of longevity risk for large annuity portfolio in Armstrong (2013).

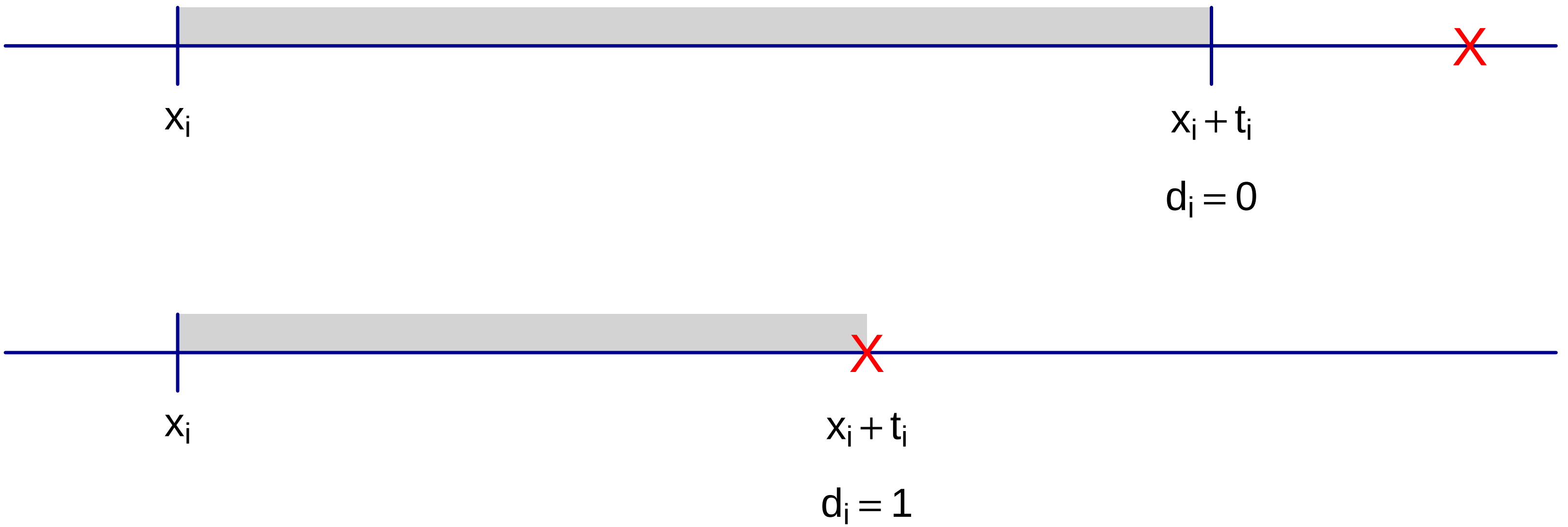
2. Mis-estimation risk

- It is also a risk in pricing, especially for block deals or reinsurance.
- Bulk annuities and longevity swaps often priced on experience data.

$\rightarrow$ Mis-estimation risk is not just a concern for the regular ICA report.

3. Mortality model

3. Survival models



Time observed, t_i, is shown in grey, while deaths are marked $\times$.

Source: Richards (2008).

3. Mortality model — structure

We use a parametric survival model for the force of mortality, $\mu_{x,y}$:

$$\mu_{x,y} = \frac{e^{\epsilon} + e^{\alpha+\beta x+\delta(y-2000)}}{1 + e^{\alpha+\beta x+\delta(y-2000)}}$$

at exact age x and time y.

Source: Richards (2014a).

3. Mortality model — individual lifetimes

Each life i has their own exact age x_i at time y_i.

$$\mu_{x_i,y_i} = \frac{e^{\epsilon} + e^{\alpha+\beta x_i+\delta(y_i-2000)}}{1 + e^{\alpha+\beta x_i+\delta(y_i-2000)}}$$

Source: Richards (2014a).

3. Mortality model — risk factors

Each life i has their own risk-factor combination, α_i:

$$\alpha_i = \alpha_0 + \sum_j \alpha_j I_{i,j}$$

where:

- α_j represents the effect of risk factor j, and
- $I_{i,j}$ is an indicator function taking the value 1 when life i possesses risk factor j and 0 otherwise.

Source: Richards (2014a).

3. Mortality model — likelihood function

The likelihood function for a survival model is as follows:

$$L \propto \prod_{i=1}^{n} {}_{t_i}p_{x_i,y_i} \times \mu_{x_i,y_i}^{d_i}$$

where d_i takes the value 0 for survivors and 1 for deaths.

3. Mortality model — log-likelihood function

In practice it is easier to work with the log-likelihood:

$$\ell = \sum_{i=1}^{n} -H_{x_i,y_i}(t_i) + d_i \log \mu_{x_i,y_i}$$

where $H_{x,y}(t)$ is the *integrated hazard function*:

$$H_{x,y}(t) = \int_0^t \mu_{x+s,y+s} ds$$

Richards (2008) tabulates $H_x(t)$ for a variety of mortality laws.

3. Generating consistent alternative parameter sets

- ℓ is a log-likelihood function of some data and a parameter vector, $\underline{\theta}$.
- Assume ℓ twice differentiable with joint maximum-likelihood vector, $\hat{\underline{\theta}}$.
- Let $\mathcal{H}$ be the matrix of second-order partial derivatives of ℓ at $\hat{\underline{\theta}}$.
- Let $\mathcal{I} = -\mathcal{H}$ be the observed information matrix.
- Consistent alternative parameter sets can be obtained by simulating from MVN$(\hat{\underline{\theta}}, \mathcal{I}^{-1})$.

Source: Richards (2014), Section 5.

4. Quantifying mis-estimation risk

4. Quantifying mis-estimation risk

"Mis-estimation risk lends itself to statistical analysis if there is sufficient accurate data"

Armstrong (2013)

"The impact of uncertainty should always be quantified financially."

Makin (2008)

4. Quantifying mis-estimation risk

- Basic procedure:
    1. Fit a parametric survival model to portfolio's experience data.
    2. Use the covariance matrix to generate alternative parameter sets.
    3. Value in-force liabilities using the alternative parameter sets.
    4. Collect liability valuations into set, S.

- S is a sample of the distribution of financial impact of mis-estimation.
- S can then be analysed to understand mis-estimation risk...

4. Quantifying mis-estimation risk

- Let S_p be the p^{th} quantile of S.
- Then:
  - $\rightarrow$ $S_{0.5}$ is the median or central liability.
  - $\rightarrow$ $S_{0.025}$ and $S_{0.975}$ give a 95% confidence interval for the liability.
  - $\rightarrow$ $S_{0.995}$ is the 99.5% ICA/Solvency II stressed liability.

- In practice we quote the mis-estimation capital as:

$$\left(\frac{S_{0.995}}{S_{0.5}} - 1\right) \times 100\%$$

- Can use mean of S in place of $S_{0.5}$ (difference is usually negligible).

4. Quantifying mis-estimation risk

"a small fund would wish (or, failing which, be required) to hold a proportionately larger opening mortality margin than a large fund, all else being equal."

Makin (2008)

4. Impact of size of data set

- 99.5% mis-estimation capital as percentage of best-estimate reserve:

Data set	Date range of data	Number of lives	Mis-estimation capital
UK pensioners	2007–2012	15,698	4.4–4.7%
German pensioners	2007–2011	244,908	1.1–1.2%

→ Larger portfolios with more data need less mis-estimation capital.

Source: Richards (2014), pages 15 and 17. The same model structure is fitted to each data set and bootstrapping confirmed the broad financial suitability of each model. The larger data set can support a richer model, however, as shown in Richards et al (2013).

5. Rationale for methodology

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Q1. Why use a parametric statistical model?

Q2. Why use the covariance matrix to create alternative parameter sets?

Q3. Why use a full portfolio valuation?

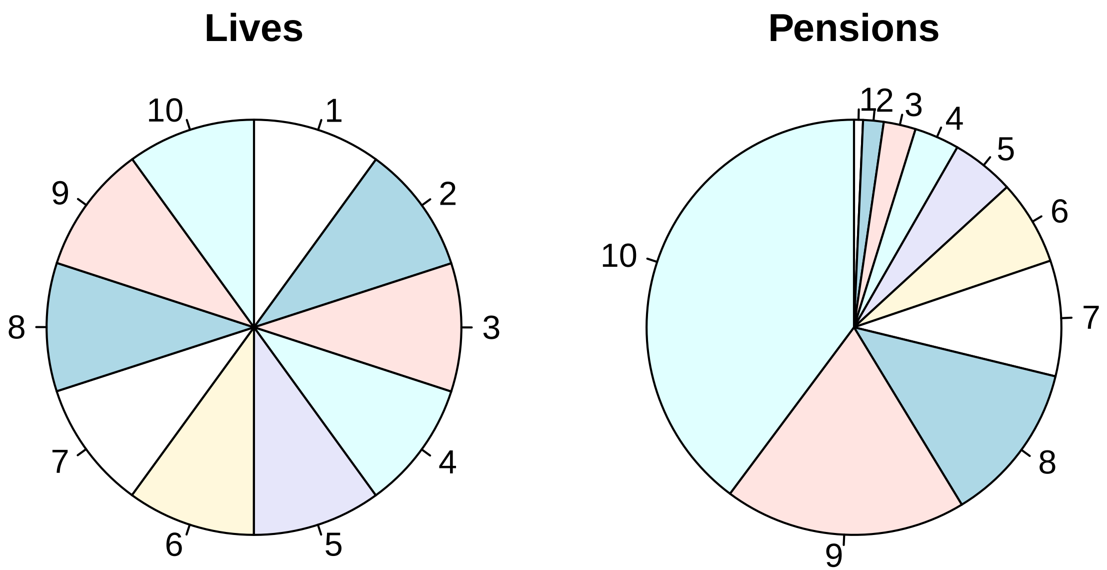
5. Case study

- Local authority pension scheme in England & Wales.
- Interested in buy-out or longevity swap.
- Pensions in payment only.
- 17,067 records, of which 2,265 are historic deaths.
- De-duplication identifies 16,131 people behind 17,067 pensions.

Source: Richards (2014), Appendix I.

5. Case study — concentration of risk

- Top tenth of pensioner population receives 39.8% of all pensions:



- . . . and next two tenths of pensioner population receive further 31.4%.

Source: Data from Richards (2014), Appendix I.

5. Case study — concentration of risk

- Parameter estimates for model with age, gender and pension size:

Parameter	Estimate	Std. Err	Lives
Age	0.148	0.005	15,698
Gender.M	0.479	0.060	5,956
Intercept	-14.731	0.491	15,698
Makeham	-5.420	0.154	15,698
Pension size — medium	-0.180	0.078	3,140
Pension size — largest	**-0.313**	**0.108**	**1,567**
Time	-0.046	0.016	15,698

→ Lives with largest pensions have lowest mortality, but estimate also has greater uncertainty.

Source: Richards (2014), Table 6.

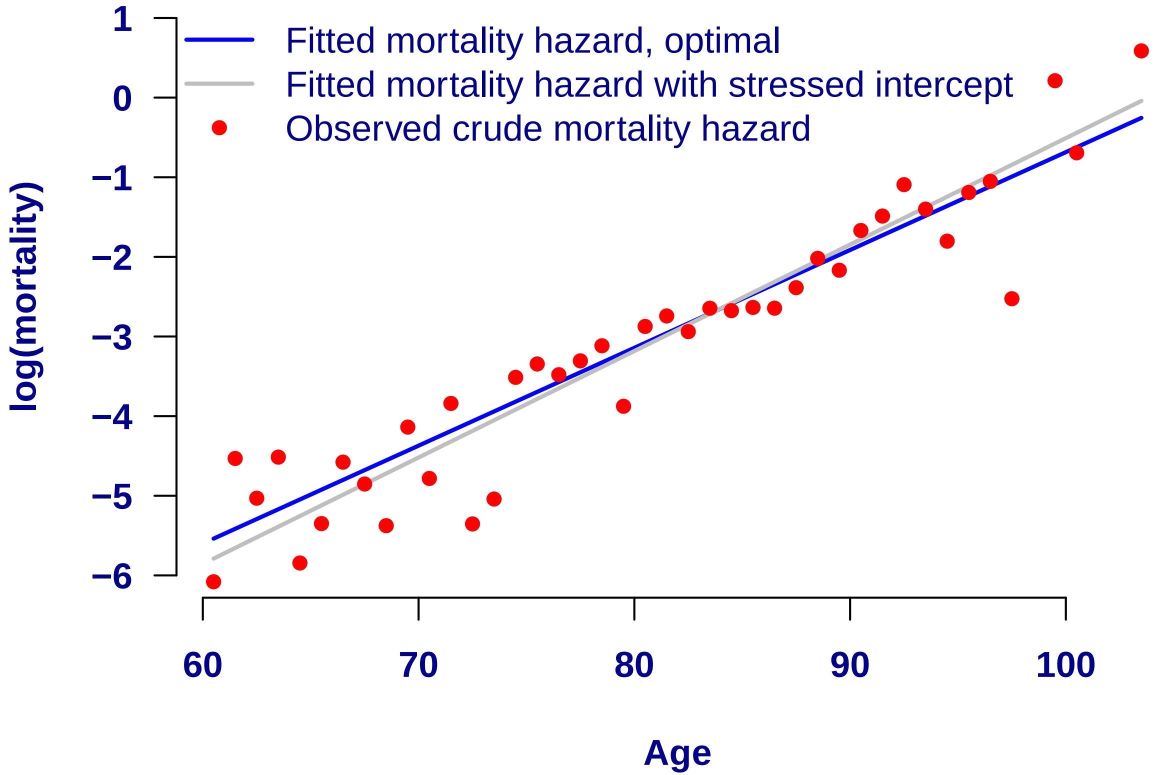
5. Rationale for methodology

Q1. Why use a parametric statistical model?

A1. Liabilities concentrated in subgroups with different characteristics.

5. Rationale for methodology — correlations

- Mortality level and slope are highly negatively correlated:



Source: Richards (2014), Figure 4.

5. Rationale for methodology — correlations

- Mortality level and slope are highly negatively correlated:

$\rightarrow$ A downward stress on the level causes an increase in the slope.

5. Rationale for methodology — correlations

- Other relevant parameter values are also correlated:
  - Age and effect of being male: +23% correlation.
  - Largest pension and effect of being male: -19% correlation.

Source: Selected correlations from Table 9 in Richards (2014).

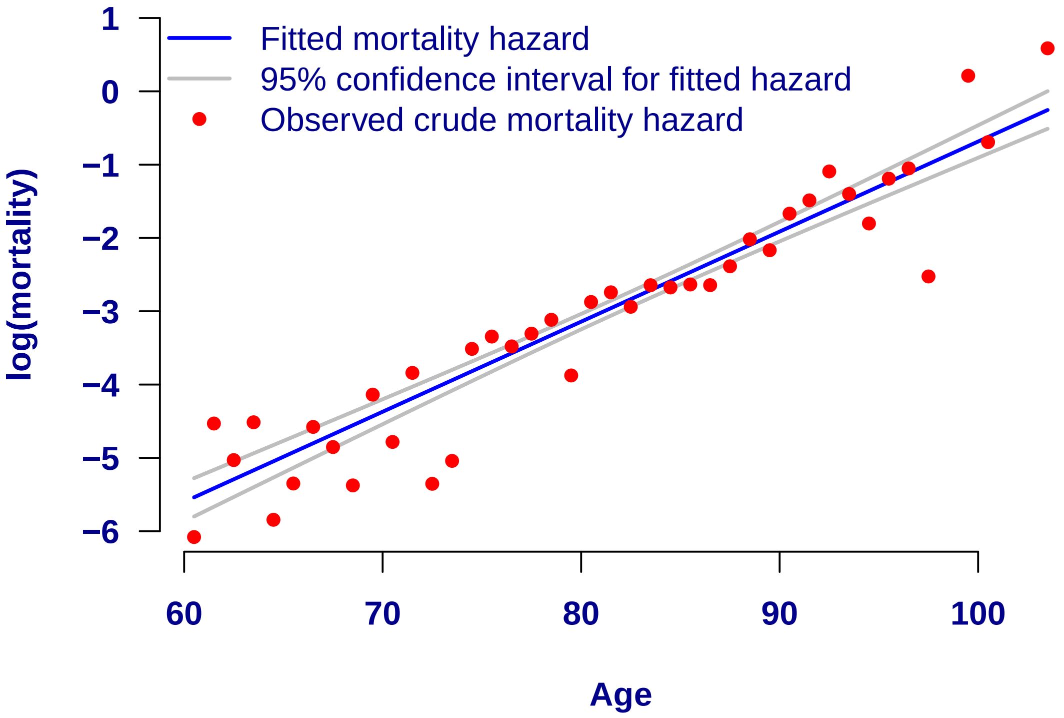
5. Rationale for methodology — correlations

Q2. Why use the covariance matrix to create alternative parameter sets?

A2. Parameters are correlated in complex ways and vary by age.

5. Rationale for methodology — liability profile

- Uncertainty varies by age:

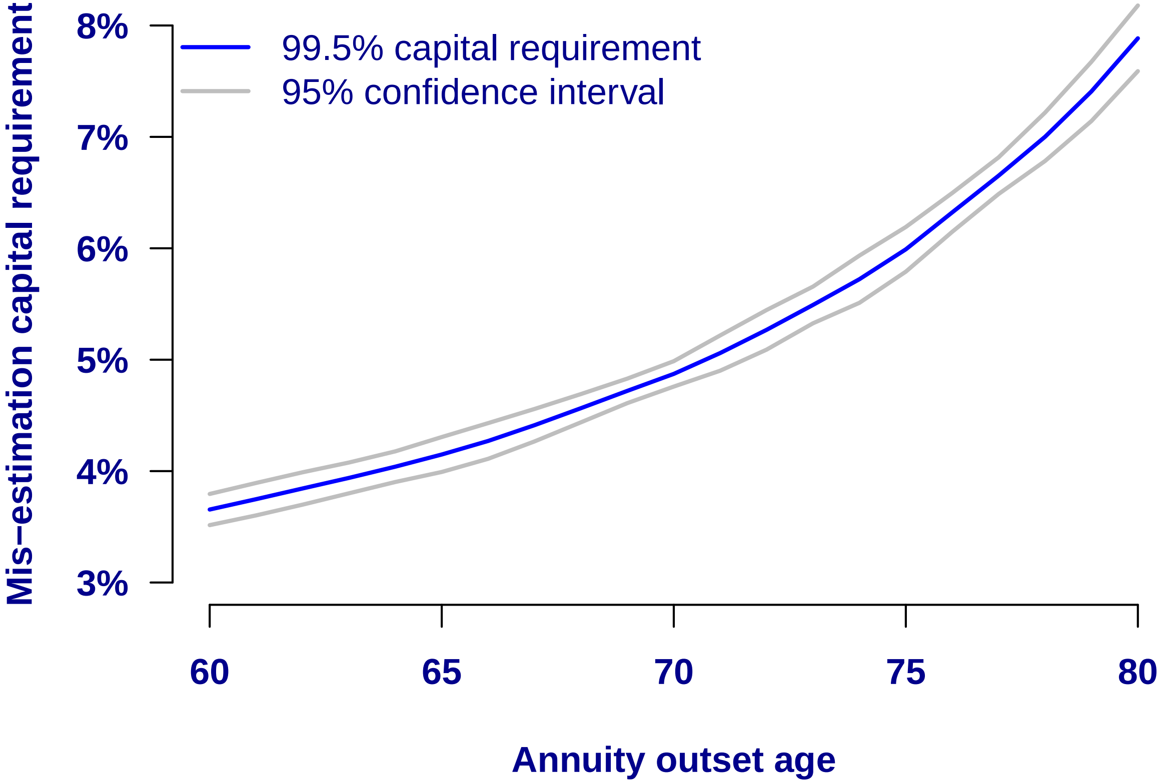


→ Uncertainty greatest at ends of age interval and least in middle.

Source: Richards (2014), Figure 3.

5. Rationale for methodology — liability profile

- Mis-estimation capital varies by age:



Source: Richards (2014), Figure 5.

5. Rationale for methodology — liability profile

Q3. Why use a full portfolio valuation?

A3. Mis-estimation impact varies by age and individual characteristics.

5. Rationale for methodology — summary

- The approach outlined here:
  - — allows for concentration of risk.
  - — allows for multiple risk factors.
  - — allows for differing levels of uncertainty over risk factors.
  - — allows for correlations between risk factors.
  - — allows for varying uncertainty by age.
  - — allows for liability profile.

6. Expressing results

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"An opening mortality experience assumption is perhaps most naturally presented in the context of a statistical confidence interval surrounding it, e.g. [x]% of mortality table, plus or minus [y]%, with [z]% certainty."

Makin (2008)

6. Expressing results

- Often need to express results in terms of a published reference table.
- Solution is to solve the following:

$$\sum_{i=1}^{n} w_i \ddot{a}_{x_i}^{T} = S_p$$

where S_p is the appropriate percentile of S and w_i is the annual pension paid to the life aged x_i.

- T is target basis, e.g. the percentage of the published reference table which equates to the relevant quantile.

Source: Richards (2014), Appendix 4.

6. Expressing results

- Solving for $S_{0.5}$ gives the best-estimate basis:

  e.g. 88.5%/87.2% of S2PA for males/females in case study.

- Solving for $S_{0.025}$ and $S_{0.975}$ gives a 95% confidence interval:

  e.g. (78.7%, 99.5%) of S2PA for males in case study.

  (note that interval is not symmetric about the best-estimate)

- Solving for $S_{0.995}$ gives 99.5% ICA/Solvency II mis-estimation stress:

  e.g. 76.0%/77.0% of S2PA for males/females in case study.

Source: Richards (2014), Table 15.

7. Preconditions

7. Preconditions

"What assumptions are you making, e.g. independence? Duplicate policies? Amounts vs lives?"

Armstrong (2013)

7. Preconditions

- There are four preconditions for the results of this method to be valid:

(i) Independent observations.
(ii) Bootstrap results close to 100%.
(iii) Quadratic log-likelihood.
(iv) Applicability of model.

- Failure of any one of these will under-estimate mis-estimation risk.

7. Preconditions — (i) independent observations

- The assumption of independence *must* hold.
- A person cannot appear more than once in the data.
- If this is not enforced, misestimation risk will be underestimated.

7. Preconditions — (i) independent observations

- Independence comes from doing one thing and avoiding another:
  - Do: deduplicate during data preparation.
  - Don't: split up an individual's exposure time for a q_x GLM[1].

[1] See http://www.longevitas.co.uk/site/informationmatrix/logisticalnightmares.html

7. Preconditions — (i) independent observations

- Why shouldn't you split up exposure times for a q_x GLM?
- Consider a life aged 70 observed for three years.
- It is invalid to treat this as three binomial trials with probabilities (q_{70}, q_{71}, q_{72}).
- Reason is that the very existence of the last trial determines what the preceding results must be.
- The pattern (Death, Survival, Death) is obvious nonsense, but it has a positive probability under the binomial GLM.

7. Preconditions — (ii) bootstrap results

- Bootstrapping is repeated sampling to test model's predictive ability.
- For each sample, compare actual deaths to predicted deaths.
- Calculate ratio on lives and amounts-weighted basis.
- A ratio close to 100% by both lives and amounts is desired...

7. Preconditions — (ii) bootstrap results

- Two alternative models fitted to same case-study experience data:

Risk factors in model	Median bootstrap ratio: (i) Lives	(ii) Amounts	Mis-estimation capital
Age+Gender+Time	99.8%	**91.7%**	3.8%
Age+Gender+Time+Size	99.8%	98.8%	4.5%

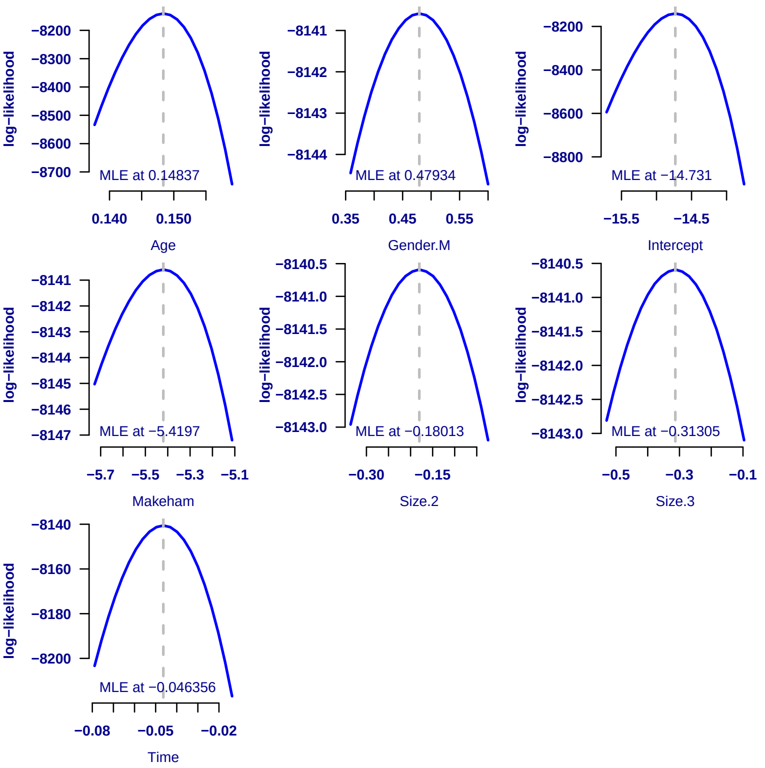
- First model is unsuitable for financial use because amounts-weighted bootstrap ratio is too far from 100%.

→ First model therefore underestimates mis-estimation risk.

Source: Table 5 in Richards (2014).

7. Preconditions — (iii) quadratic log-likelihood

- Using the covariance matrix only works if log-likelihood is quadratic:



7. Preconditions — (iv) applicability

- Judgement will often be required.
- Example of recent change in distribution strategy: does a model calibrated to existing data adequately capture new risks?

→ Mis-estimation capital from this approach will often be lower bound.

8. Conclusions

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- Mis-estimation risk stems from having finite data.
- Mis-estimation risk should be assessed not only for ICA and Solvency II, but also for pricing large block deals.
- Quantification must be financial and account for both correlations and concentration of risk.
- Larger portfolios tend to have less uncertainty and thus lower mis-estimation capital.
- Ensure observations are independent and bootstrap ratios close to 100%.

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