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The Hermite-spline model of mortality

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2. Curve plotting
3. Hermite mortality model
4. Selection effects
5. Time trend
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1 Parametric mortality models

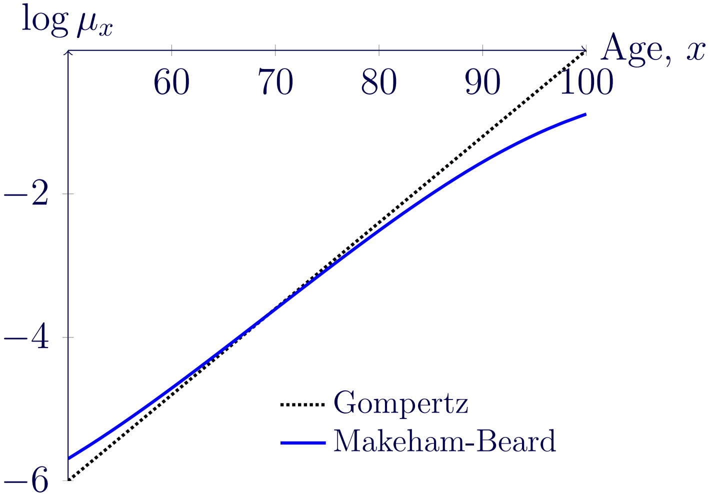
1 Parametric mortality models

Gompertz [1825] $\mu_x = e^{\alpha+\beta x}$

Makeham-Beard $\mu_x = \dfrac{e^{\epsilon} + e^{\alpha+\beta x}}{1 + e^{\alpha+\rho+\beta x}}$

See Richards [2012] for more extensive list.

1 Parametric mortality models



1 Parametric survival models

Role of parameters:

- α shifts level of $\log \mu_x$ up or down.
- β describes rate of change of $\log \mu_x$ by age.

α and β strongly correlated [Richards et al., 2013].

1 Parametric mortality models

Each life i gets its own personal values of α and β:

$$\alpha_i = \alpha_0 + \sum_{j=1}^{m} \alpha_{r_j} z_{i,j} \tag{1}$$

$$\beta_i = \beta_0 + \sum_{j=1}^{m} \beta_{r_j} z_{i,j} \tag{2}$$

- α_0 is $\log \mu_0$ for baseline group,
- β_0 is the baseline rate of increase with age x,
- α_{r_j} is the main effect of risk factor $r_j, j \in \{1, 2, \ldots, m\}$,
- β_{r_j} is the interaction of age with risk factor r_j.
- $z_{i,j} = 1$ if risk factor r_j applies, zero otherwise.

Three problems with β_{r_j}:

1. Crossover.
2. Redundancy.
3. Signal strength.

1 Crossover

Mortality crossover in $\log \mu_x$ for Gompertz model fitted to lives in a UK pension scheme. Source: Macdonald et al. [2018, p120].

1 Crossover

- Do wealthier pensioners really have higher mortality above age 85?
- Data don't support this; it is a model artefact.
- Not prudent above age 85...

  ...but model fit also distorted at younger ages.

1 Redundancy

- Mortality differentials vanish around age 95.
- β_{r_j} has opposite sign to α_{r_j} (mortality convergence).
- Could replace β_{r_j} with $\dfrac{-\alpha_{r_j}}{95}$ (or similar).

  $\Rightarrow$ Don't strictly need β_{r_j} (redundancy).

1 Signal strength

- Detecting a main effect (α_{r_j}) is easier than detecting variation of effect by age (β_{r_j}).

  $\Rightarrow$ can be hard to reliably estimate $\beta_{r_j}\ldots$

  ….despite strong prior expectations of its value.

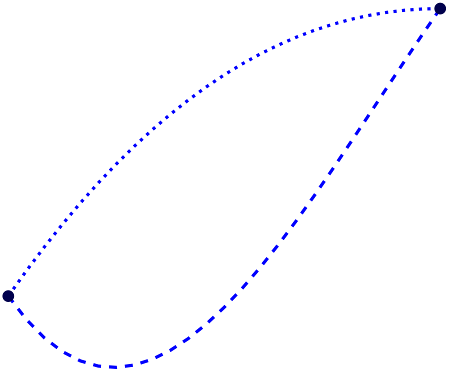
1 Wish list for a model

- Convergence of differentials by age.
- No crossover.
- No redundant parameters.

2 Curve plotting

2 Curve plotting

In computer graphics we often want to draw a smooth path from one point to another:



2 Charles Hermite (1822-1901)

2 Hermite splines

Hermite basis functions:

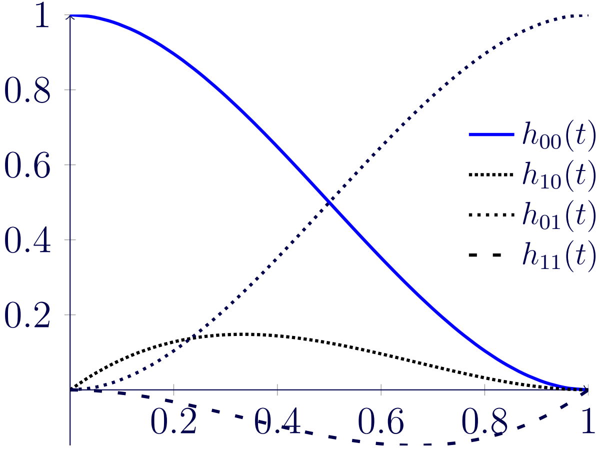
$$h_{00}(t) = (1 + 2t)(1 - t)^2 \quad (3)$$

$$h_{10}(t) = t(1 - t)^2 \quad (4)$$

$$h_{01}(t) = t^2(3 - 2t) \quad (5)$$

$$h_{11}(t) = t^2(t - 1) \quad (6)$$

2 Hermite splines

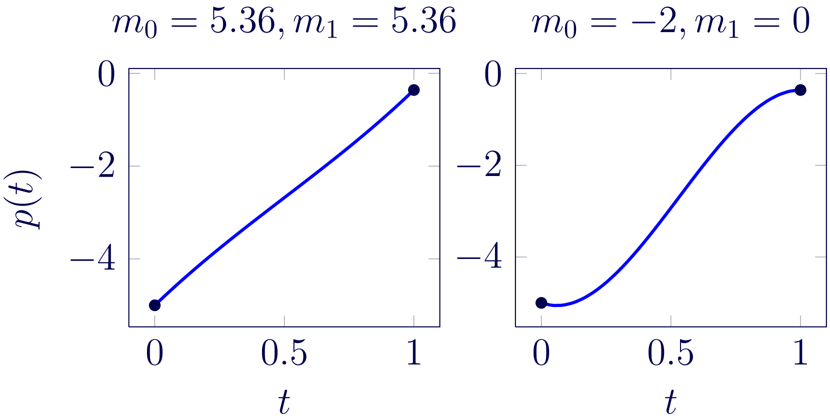


2 Hermite splines

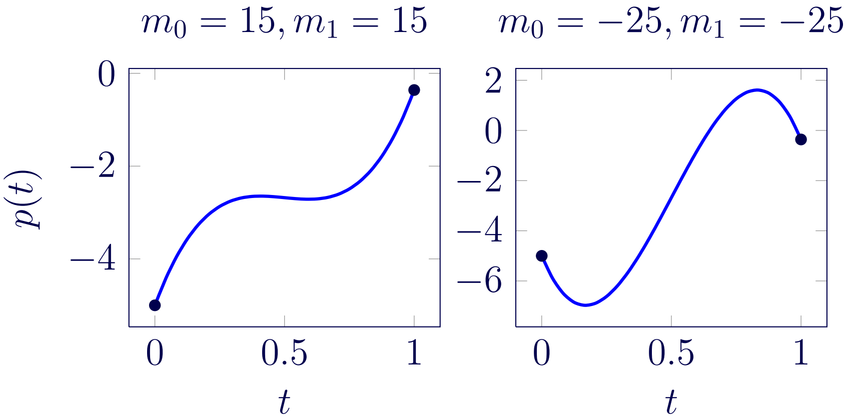
Smooth curve drawn as set $\{(t, p(t))\}$, where:

$$p(t) = p_0 h_{00}(t) + m_0 h_{10}(t) + p_1 h_{01}(t) + m_1 h_{11}(t) \quad (7)$$

2 Hermite splines



2 Hermite splines



2 Hermite splines

What does this have to do with mortality modelling?

3 Hermite mortality model

3 Hermite mortality model

- Set age interval $[x_0, x_1]$.
- Set $p_0 = \log \mu_{x_0}$.
- Set $p_1 = \log \mu_{x_1}$.
- Set $t = \dfrac{x - x_0}{x_1 - x_0}$.

  ... then $\{(t, p(t))\}$ will trace $\log \mu_x$ in $[x_0, x_1]$.

3 Hermite mortality model

Smooth curve drawn as $\{(t, p(t))\}$, where:

$$p(t) = p_0 h_{00}(t) + m_0 h_{10}(t) + p_1 h_{01}(t) + m_1 h_{11}(t) \quad (8)$$

Logarithm of force of mortality, μ_x:

$$\log \mu_x = \log \mu_{x_0} h_{00}(t) + m_0 h_{10}(t) + \log \mu_{x_1} h_{01}(t) + m_1 h_{11}(t) \tag{9}$$

3 Hermite mortality model

Model for $\log \mu_x$:

$$\log \mu_x = \alpha h_{00}(t) + m_0 h_{10}(t) + \omega h_{01}(t) + m_1 h_{11}(t) \quad (10)$$

3 Hermite mortality model

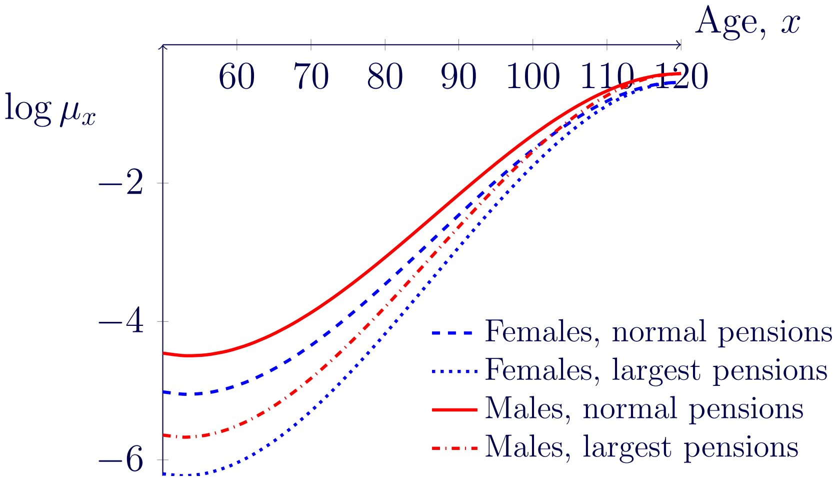
Interactive online demo:

www.longevitas.co.uk/site/Hermite/HermiteAge.html

3 Hermite mortality model

- Estimate α, ω, m_0 and m_1.
- α_i for life i structured as before.
- *However*, α_i is modulated by reducing function h_{00}.
  $\Rightarrow$ effect of risk factors reduces with age (convergence).
- h_{00} never changes sign.
  $\Rightarrow$ no crossover.
- No β term.
  $\Rightarrow$ no redundant parameters.

3 Gender and pension size



Source: Richards [2019].

3 Hermite mortality model

- ✓ Convergence of differentials by age.
- ✓ No crossover.
- ✓ No redundant parameters.

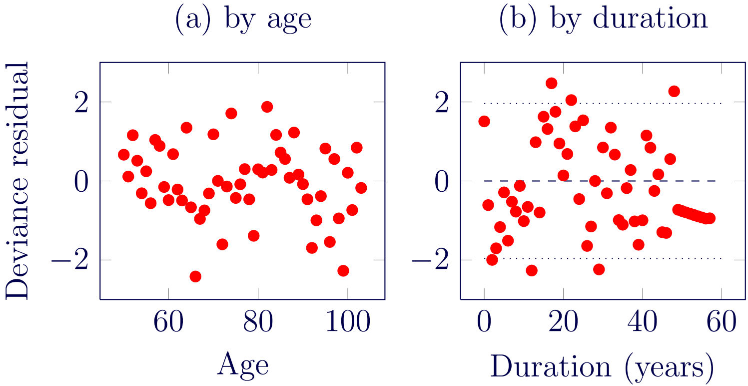
What more could we ask?

4 Selection effects

4 Selection effects

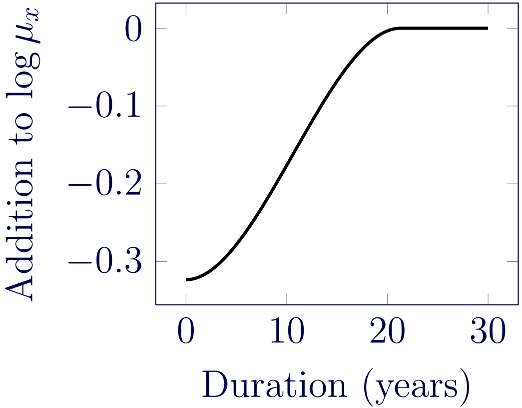
- Mortality can also vary by since contract start.
- Are there selection effects amongst pensioners?

Deviance residuals for model with age, gender and pension size:



Source: Richards [2019].

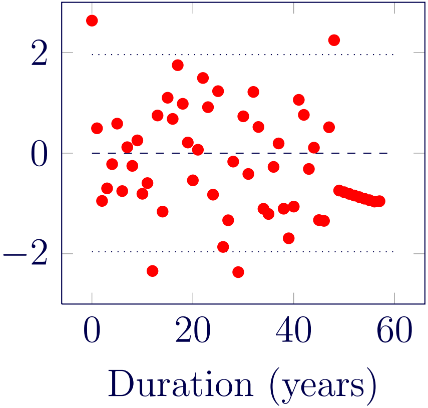
Addition to $\log \mu_x$ in respect of for selection effects:



Source: Richards [2019].

4 Actual portfolio selection

Deviance residuals by duration:



Source: Richards [2019].

5 Time trend

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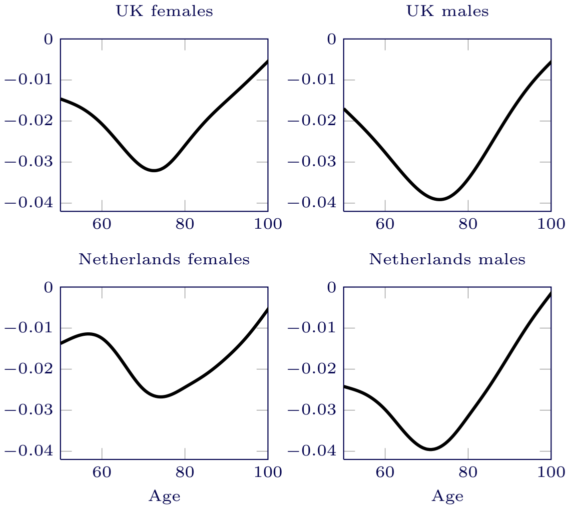
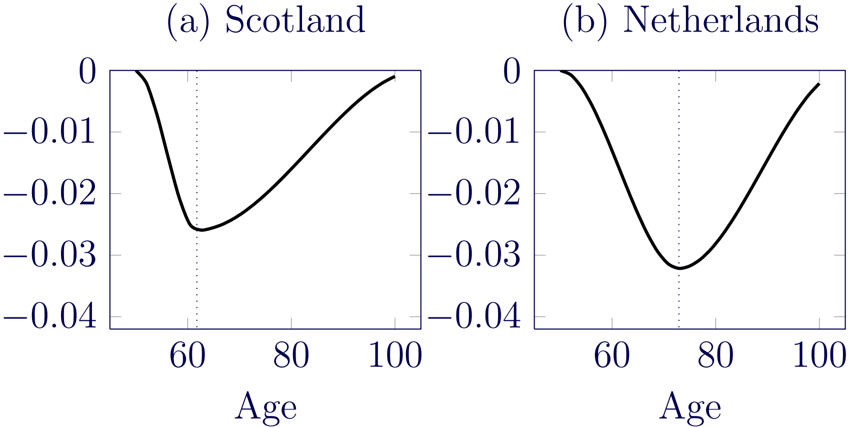
- Mortality also varies by time.
- Time trend often age-dependent...

5 Population time trends



Source: Richards [2019].

5 Actual portfolio time trends



Source: Richards [2019].

6 Seasonal variation

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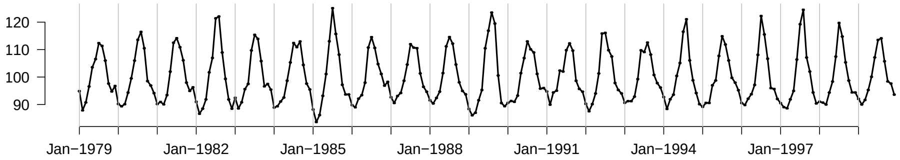
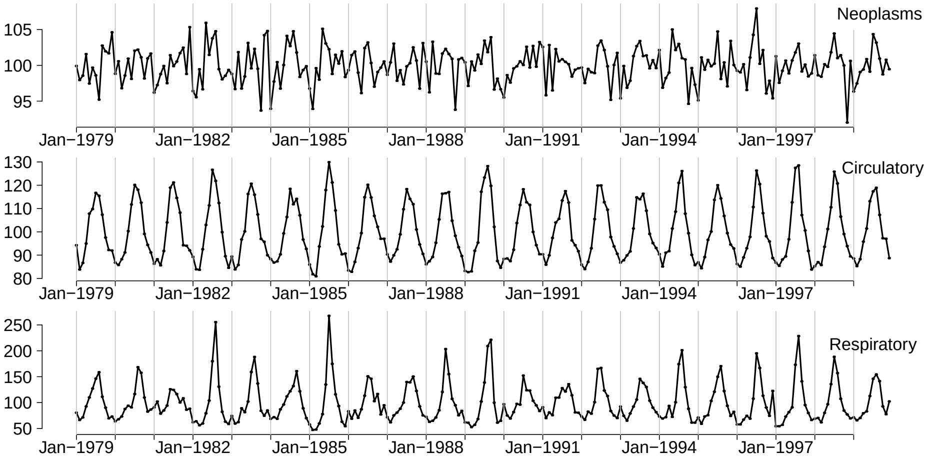
Percentage of average daily number of deaths in Australia:



Source: de Looper [2002]. All causes, 1979–1999.

6 Seasonal variation

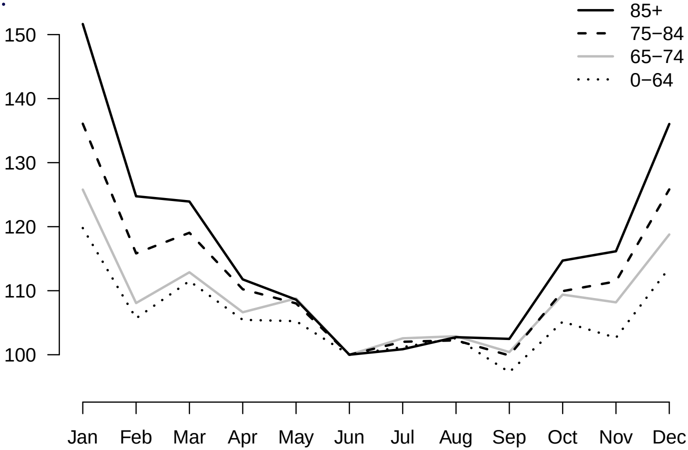
Percentage of average daily number of deaths in Australia:



Selected causes, 1979–1999. Source: de Looper [2002].

6 Seasonal variation

Deaths in England and Wales as percentage of June count, 2015–17:



Source: Richards [2019].

6 Seasonal variation

Add a cyclic factor for the mortality hazard:

$$\log \mu^*_{x,r,y} = \log \mu_{x,r,y} + e^\zeta \cos\left(2\pi(y-\tau)\right) \qquad (11)$$

- τ is proportion of the year after January 1st when mortality peaks
- e^ζ is the peak additional mortality at that time (on logarithmic scale).

6 Portfolio seasonal variation

Country	Portfolio nature	$\hat{\zeta}$	$\hat{\tau}$	Peak mortality: (a) as % of average	Peak mortality: (b) time of year
Scotland	Pension plan	-1.62	0.092	122%	Feb 1st
UK	Insurer annuities	-2.00	0.001	114%	Dec 30th
England	Pension plan	-2.02	0.071	114%	Jan 25th
Netherlands	Pension plan	-2.25	0.055	111%	Jan 19th
England	Pension plan	-2.29	0.048	111%	Jan 16th
UK	Insurer annuities	-2.27	0.086	111%	Jan 30th

Source: Richards [2019].

7 The bottom line

7 Impact of risk factors

Change in discounted cashflow valuation from adding risk factors. Period rates until the model in the last line of the table.

Model	AIC	Change in AIC	PV (£m): Males	Females	Total	Change in total PV
Hermite I, age only	27279.4	n/a	532.7	302.6	835.3	n/a
+gender	27192.5	−86.9	495.0	321.4	816.4	−18.9
+widow(er) status	27177.6	−14.9	495.7	328.6	824.3	7.9
+early-retirement status	27161.7	−15.9	485.6	324.6	810.2	−14.1
+pension size	27099.5	−62.2	538.5	333.6	872.1	61.9
+selection	27082.1	−17.4	534.7	328.3	863.0	−9.1
+season	27045.8	−36.3	532.3	326.8	859.1	−3.9
change from period mortality to forecast mortality:						
+age-related time trend	27044.0	−1.8	579.5	352.9	932.4	73.3

Source: Richards [2019].

7 The bottom line

- 8 risk factors modelled with just 14 parameters.

  $\Rightarrow$ model is very parsimonious.
- Risk factors vary in statistical and financial significance:
  - Least statistically significant (time trend) is very significant financially.
  - Least financially significant (season) is very significant statistically.

8 Conclusions

8 Conclusions

- Hermite splines offer flexible modelling of $\log \mu_x$.
- Long list of benefits:
  - Automatic convergence of mortality differentials by age.
  - No crossover.
  - No redundant parameters.
  - Fewer parameters.
  - Selection effects.
  - Age-modulated time trend.
  - Seasonal variation.

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More on longevity risk at www.longevitas.co.uk

9 Legal matters

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